

A Framework For Constructing and Evaluating Mathematical Tasks

Guillermo Bautista, Jr.

National Institute for Science and Mathematics

Education Development (UP NISMED)

University of the Philippines, Philippines

Mathematical Tasks

- A mathematical task is a set of problems or a single complex problem that focuses students' attention on a particular mathematical idea (Stein, Grover, Henningsen, 1996)
- 80% of the time are spent on tasks (Hiebert, et. al. 2003)
- Mathematical tasks influence student learning (Doyle, 1988)

FRAMEWORK (Cognitive Demand)

- Memorization, procedures without connections, procedures with connections, doing mathematics (Stein, 1998)
- Memorize; perform procedures; communicate understanding; solve non-routine problems; conjecture, generalize, prove (Porter, 2002)
- Low, moderate, high complexity items (NAEP, 2009)
- Knowing, Applying, Reasoning (TIMSS, 2015)

FRAMEWORK (Solutions/Answers)

- Single/multiple computation procedures; numerical answer, numerical expression, explanation/solution (Li, 2000)
- Mathematical expressions, generating examples, graphical modeling (Bennet, Borley, and Quardt, 2000)
- Traditional, non-traditional; open-ended, close-ended; routine, non-routine (Zhu & Fan, 2006)
- Type 1: one solution, one answer; Type 2: many solutions, one answer; Type 3: many solutions, many answers (Isoda & Katagiri, 2012)

TIMSS

The Trends in International Mathematics and Science Study (TIMSS) is a series of international assessments of the mathematics and science studies around the world.

TIMSS classifies tasks into Knowing, Applying, and Reasoning.

Knowing

Recall	Recall definitions, terminology, number properties, units of measurement, geometric properties, and notation (e.g., $a \times b = ab$, $a + a + a = 3a$).
Recognize	Recognize numbers, expressions, quantities, and shapes. Recognize entities that are mathematically equivalent (e.g., equivalent familiar fractions, decimals, and percents; different orientations of simple geometric figures).
Classify/Order	Classify numbers, expressions, quantities, and shapes by common properties.
Compute	Carry out algorithmic procedures for $+$, $-$, \times , \div , or a combination of these with whole numbers, fractions, decimals, and integers. Carry out straightforward algebraic procedures.
Retrieve	Retrieve information from graphs, tables, texts, or other sources.
Measure	Use measuring instruments; and choose appropriate units of measurement.

Knowing Tasks

Grade School

Which of these fractions is larger than $\frac{1}{2}$?

A. $\frac{3}{5}$

B. $\frac{3}{6}$

C. $\frac{3}{8}$

D. $\frac{3}{10}$

High School

Solve this inequality.

$$9x - 6 < 4x + 4$$

Answer: _____

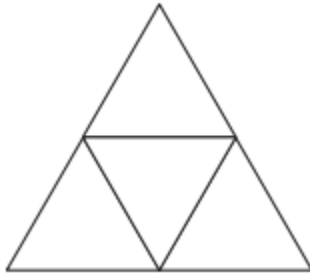
Applying

Determine	Determine efficient/appropriate operations, strategies, and tools for solving problems for which there are commonly used methods of solution.
Represent/Model	Display data in tables or graphs; create equations, inequalities, geometric figures, or diagrams that model problem situations; and generate equivalent representations for a given mathematical entity or relationship.
Implement	Implement strategies and operations to solve problems involving familiar mathematical concepts and procedures.

Applying Tasks

Grade School

Shade $\frac{1}{2}$ of the large triangle.



High School

$(0, -1)$, $(1, 3)$

Which equation is satisfied by BOTH of these pairs of numbers (x, y) ?

- A. $x + y = -1$
- B. $2x + y = 5$
- C. $3x - y = 0$
- D. $4x - y = 1$

The average age of six people in a room is 30 years. A 20-year-old person leaves the room. What is the average age of the 5 remaining people in years?

Reasoning

Analyze	Determine, describe, or use relationships among numbers, expressions, quantities, and shapes.
Integrate/Synthesize	Link different elements of knowledge, related representations, and procedures to solve problems.
Evaluate	Evaluate alternative problem solving strategies and solutions.
Draw Conclusions	Make valid inferences on the basis of information and evidence.
Generalize	Make statements that represent relationships in more general and more widely applicable terms.
Justify	Provide mathematical arguments to support a strategy or solution.

TIMSS (2015)

Reasoning Tasks

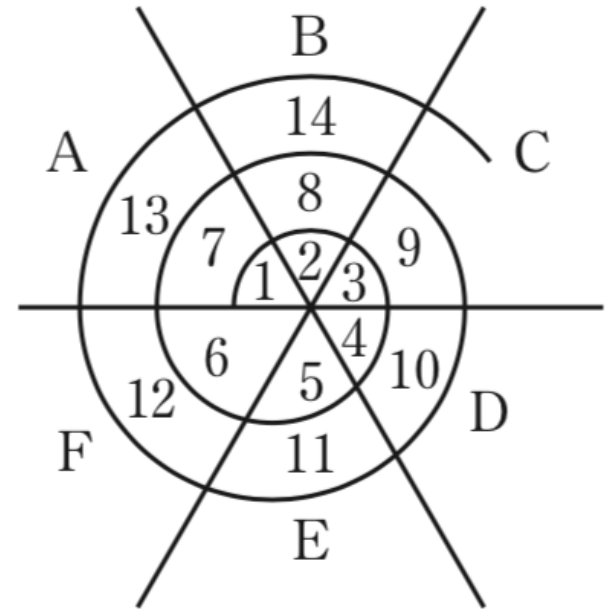
Place the four digits 3, 5, 7, and 9 into the boxes below in the positions that would give the greatest result when the two numbers are multiplied.

$$\begin{array}{r} \square \square \\ \times \square \square \\ \hline \end{array}$$

Reasoning

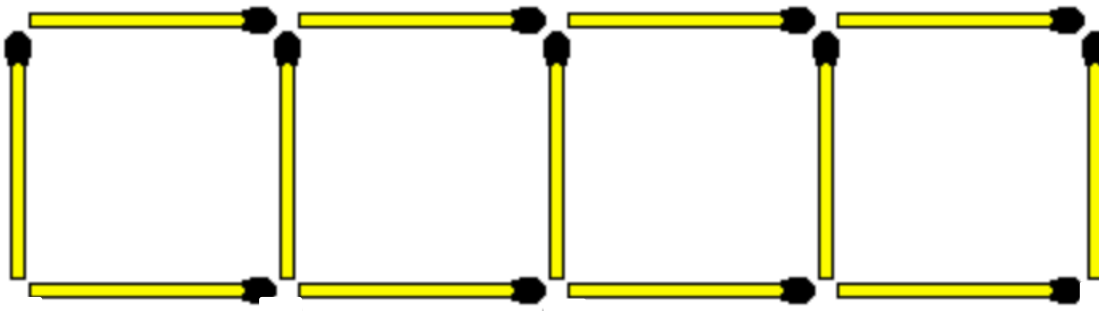
Natural numbers are written on a spiral as shown on the right divided into sections A-F.

- a.) In which section will 1000 be written?
- b.) The sum of a number from section B and a number in section E will be a number in section A. Explain this using algebraic expressions.



Reasoning

In the figure, 13 matches were used to make 4 squares in a row. What is the number of squares in a row that can be made in this way using the. 73 matches. Show the calculations that lead to your answer.



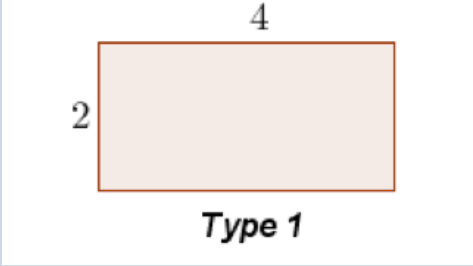
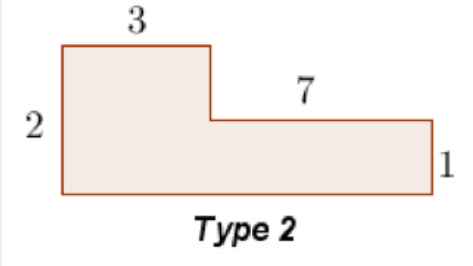
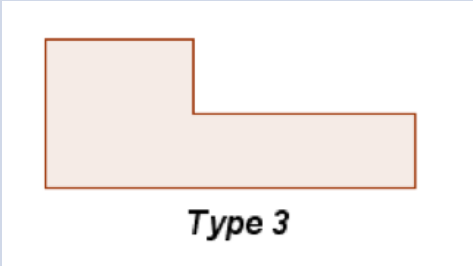
Isoda & Katagiri's Classification

Type 1: One solution, one answer

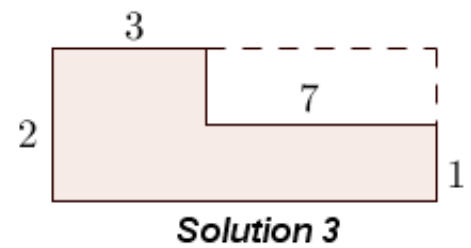
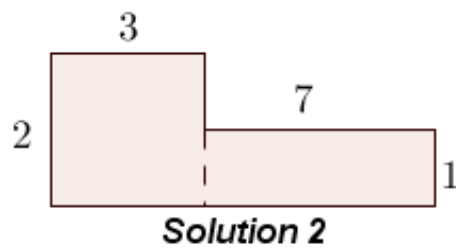
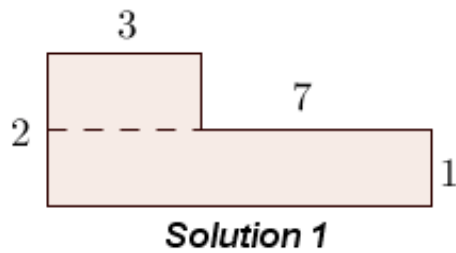
Type 2: Many solution, one answer

Type 3: Many solutions, many answers

What is the area of the figure?

Classification	Example	Description
Type 1		One solution One answer
Type 2*		Many solutions One answer
Type 3		Many solutions Many answers

Solution to *



Isoda & Katagiri's Framework (cont.)

Type 1

$$\begin{array}{r} \boxed{1} \boxed{0} \boxed{0} \\ - \quad \boxed{9} \boxed{8} \\ \hline \quad \boxed{} \end{array}$$

Type 2

$$\begin{array}{r} \boxed{} \boxed{} \boxed{} \\ - \quad \boxed{} \boxed{} \\ \hline \quad \boxed{3} \end{array}$$

Type 3

$$\begin{array}{r} \boxed{} \boxed{} \boxed{} \\ - \quad \boxed{9} \boxed{8} \\ \hline \quad \boxed{} \end{array}$$

The Bifocal Framework

Cognitive Demand

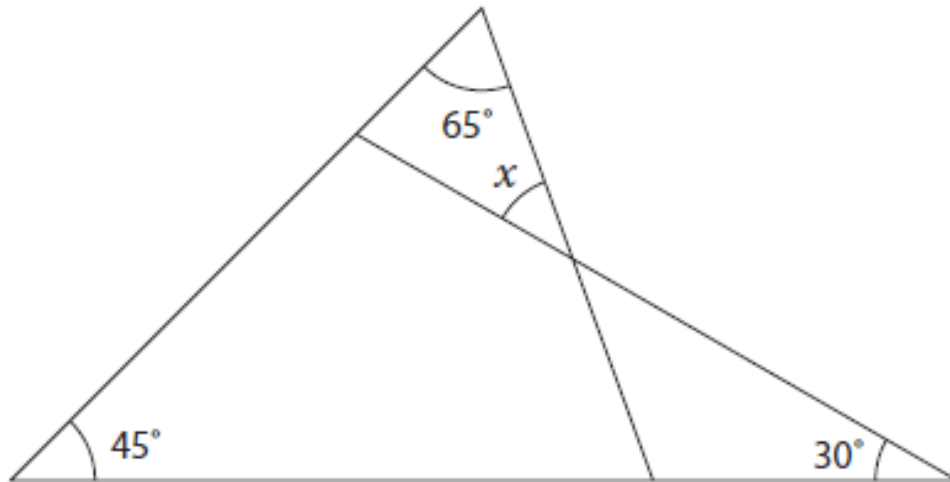
	K	A	R
1	K1	A1	R1
2	K2	A2	R2
3	K3	A3	R3

Number of
Solutions
Answers

Bautista & Isoda (2017)

Task 1

In the figure below, what is the value of x ?



TIMSS (2011)

Task 2



Figure 1

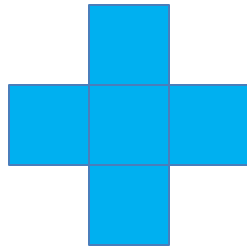


Figure 2

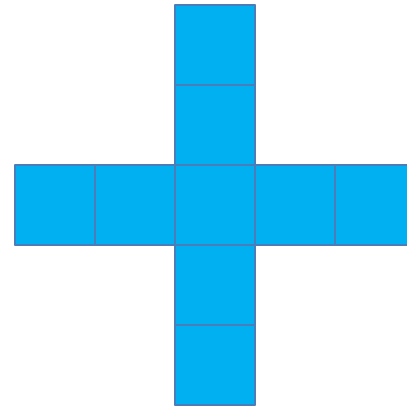


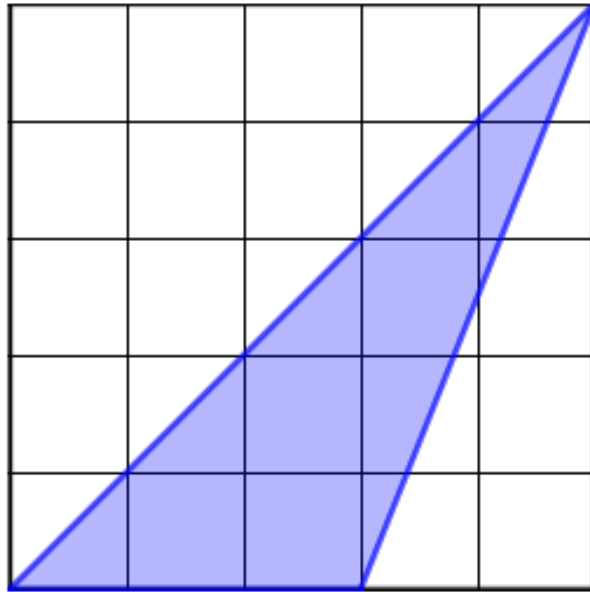
Figure 3

If the pattern continues, how many squares are there in

- a. Figure 5?
- b. Figure 10?
- c. Figure n ?

Task 3

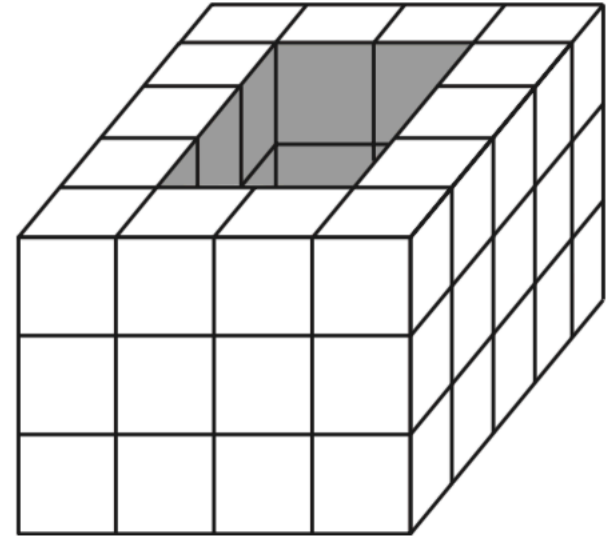
The squares in the grid below are 1cm by 1 cm. A triangle is drawn such that its vertices are on the intersection of the grid lines.



1. What is the area of the triangle?
2. Draw a triangle whose area is the same as that of the blue triangle and justify why the areas are equal.

Task 4

The figure shows a shape made up of cubes that are all the same size. There is a hole all the way through the shape. How many cubes would be needed to fill the hole?



Task 5

Use a dynamic geometry software to investigate the effects of a , h , and k to the graph of the function $f(x) = a(x - h)^2 + k$.

Task 6

Last year, Jaime's age was a multiple of 8.
This year, Jaime's age is a multiple of 11.
In how many years will he be 41 years old?

Task 7

The list of integers 4, 4, x , y , 13 has been arranged from least to greatest. How many different possible ordered pairs (x,y) are there so that the average (mean) of these 5 integers is itself an integer?

Task 8

Sally picks four consecutive positive integers. She divides each integer by four and adds the remainders together. What is the sum of the remainders?

Using the Framework

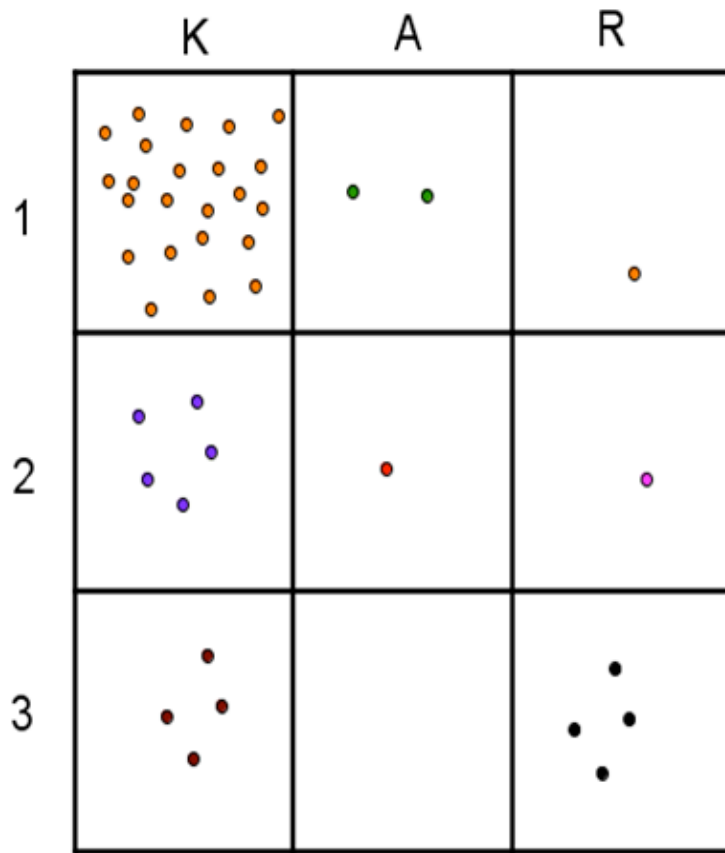
- Tasks from a Math textbook in Japan and the Philippines are compared (Division)
- Japan: Study with Your Friends Mathematics (Gakkohtosho)
- Philippines: Math Learning Materials (Department of Education)
- Tasks were classified by 2 teachers; discussion with classmates colleagues, presentation was made

Results

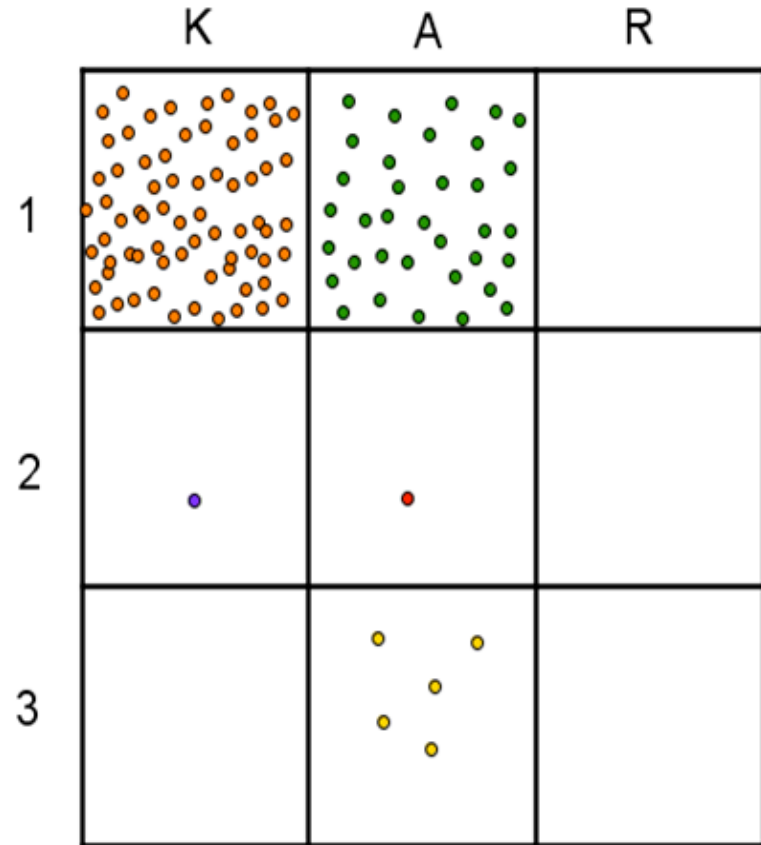
Type	Philippines (N = 112)							
	K		A		R		Total	
1	70	62.5%	35	31.3%	0	0%	105	93.8%
2	1	0.9%	1	0.9%	0	0%	2	1.8%
3	0	0.0%	5	4.4%	0	0%	5	4.4%
Total	71	63.4%	41	36.6%	0	0.0%	112	100%

Type	Japan (N = 41)							
	K		A		R		Total	
1	23	56.1%	2	4.9%	1	2.4%	26	63.4%
2	5	12.2%	1	2.4%	1	2.4%	7	17.0%
3	4	9.8%	0	0.0%	4	9.8%	8	19.6%
Total	32	78.1%	3	7.3%	6	14.6%	41	100.0%

Results



Japan



Philippines

Results

- The number of tasks on division of the Philippine textbook is three times that of Japan's textbook.
- The bulk of the tasks in the Philippine textbook are in K1 (62.5%) and A1 (31.3%).
- The mathematical tasks in the Japanese textbook were more distributed in the 2D matrix.
- The chapter on division of the Japanese textbook has more open-ended tasks compared than its Philippine counterpart. *

Conclusion

The bifocal matrix

- can be used as an analytic tool to examine the distribution mathematical tasks in textbooks
- can give a bird's eye view of the composition of the mathematical tasks
- can be used as a tool for comparing the distribution of mathematical tasks between textbooks
- can be used as a pedagogic tool for developing mathematical tasks